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Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition ☆

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Abstract

In this paper, we consider a semilinear heat equation $u_t = \Delta u + c(x,t)u^p$ for $(x,t) \in \Omega \times (0,\infty)$ with nonlinear and nonlocal boundary condition $u|_{\partial\Omega\times(0,\infty)} = \int_\Omega k(x,y,t)u^l\,dy$ and nonnegative initial data where $p > 0$ and $l > 0$. We prove global existence theorem for $\max(p,l) \leqslant 1$. Some criteria on this problem which determine whether the solutions blow up in a finite time for sufficiently large or for all nontrivial initial data or the solutions exist for all time with sufficiently small or with any initial data are also given.

Keywords: Reaction–diffusion equation; Nonlocal boundary condition; Global solution; Blow-up

1. Introduction

In this paper we consider the following nonlocal initial boundary value problem:

$$\begin{cases} u_t = \Delta u + c(x,t)u^p & \text{for } x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega k(x,y,t)u^l(y,t)\,dy & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n \geqslant 1$ with a smooth boundary $\partial\Omega$, $p > 0$ and $l > 0$. Here $c(x,t)$ is a nonnegative locally Hölder continuous function defined for $x \in \overline{\Omega}$ and $t \geqslant 0$ and $k(x,y,t)$ is a nonnegative continuous function defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$ and $t \geqslant 0$. The initial datum $u_0(x)$ is a nonnegative continuous function satisfying the boundary condition at $t = 0$.

Over the last twenty years, many physical phenomena were formulated into nonlocal mathematical models [2–4,6, 14,17,19,22]. Galaktionov and Levine [15] and Laptev [18] studied the blow-up phenomena for the Cauchy problem

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in a reaction–diffusion equation with nonlocal nonlinearities. Initial boundary value problems in a bounded domain for such equations have been considered, for example, in [1,7,12,20,21]. In particular, the authors in [1,12,20] investigated global existence and blow-up problems. Initial boundary value problem for diffusion and reaction–diffusion equations with linear boundary condition in the second equation of (1.1) has been analyzed by many authors (see, for example, [9,13,23]). Local and global existence, comparison principle and various qualitative properties, including asymptotic behavior of the solutions, have been discussed. The stability of global solutions was investigated by Deng [10,11] for a initial boundary value problem with nonlinear and nonlocal Neumann boundary condition in the Burgers' equation. Local existence and some blow-up results have been obtained also. The initial boundary value problem for semilinear Euler–Poisson–Darboux equation with a boundary condition in (1.1) for one-dimensional case was considered by Chan and Zhu [5] and some blow-up results were obtained. For the additional references, readers can see survey papers in [8,20].

Comparison principle, the uniqueness of solution with any initial data for $\min(p,l) \geqslant 1$ and with nontrivial initial data otherwise, nonuniqueness of solution with trivial initial data for $p < 1$ or $l < 1$, local existence theorem for problem (1.1) have been proved in [16]. The aim of this paper is to give some criteria for the existence of global solutions as well as for a blow-up of solutions in a finite time. Our global existence and blow-up results depend on the behavior of the coefficients $c(x,t)$ and $k(x,y,t)$ as t tends to infinity.

The plan of this paper is as follows: In Section 2, we prove global existence theorem for $\max(p,l) \leqslant 1$; general analysis of the blow-up problem we give in Section 3 and present there the nonexistence of global solutions for sufficiently large initial data, global nonexistence of all nontrivial solutions as well as global existence for sufficiently small initial data; and finally in Section 4 we study the particular cases either $p = 1$ or $l = 1$.

2. Global existence

We denote $Q_T = \Omega \times (0,T)$. In this paper, we use the following natural definition of solution, sub- and supersolution.

Definition 2.1. We say that a nonnegative function $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a subsolution of problem (1.1) in Q_T if

$$\begin{cases} u_t \leqslant \Delta u + c(x,t)u^p & \text{for } x \in \Omega,\ 0 < t < T, \\ u(x,t) \leqslant \int_\Omega k(x,y,t)u^l(y,t)\,dy & \text{for } x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) \leqslant u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{2.1}$$

and $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a supersolution if $u \geqslant 0$ and it satisfies (2.1) in the reverse order. We say that u is a solution of problem (1.1) in Q_T if it is both a subsolution and a supersolution of (1.1) in Q_T. Further, we say that u is a global solution of problem (1.1) if it is a solution of (1.1) in Q_T for any $T > 0$.

We shall also use repeatedly the notions of strict sub- and supersolution.

Definition 2.2. We say that u is a strict subsolution of the problem (1.1) in Q_T if it is a subsolution in Q_T and the second inequality in (2.1) is strict. Analogously we say that u is a strict supersolution of the problem (1.1) in Q_T if it is a supersolution in Q_T and the second inequality in (2.1) is reversed and strict.

To prove our global existence and blow-up results we need the property of positiveness of solution for $t > 0$ and $\overline{\Omega}$ and the comparison principle which have been proved in [16].

Lemma 2.3. *Let u_0 is a nontrivial function in Ω and assume that*

$$k(x,\cdot,t) \not\equiv 0 \quad \text{for any } x \in \partial\Omega \text{ and } 0 < t \leqslant T. \tag{2.2}$$

Suppose that u is the solution of (1.1) *in Q_T. Then $u > 0$ in $\overline{Q}_T$ for $t > 0$.*

Theorem 2.4. *Let v and u be a nonnegative supersolution and a nonnegative subsolution of problem* (1.1) *in Q_T, respectively, and $v(x,0) > u(x,0)$ in $\overline{\Omega}$. Suppose that either* (2.2) *holds or v is a strict supersolution. Then $v(x,t) > u(x,t)$ in $\overline{Q}_T$.*

The following theorem gives a global existence result for problem (1.1) and the proof of this theorem relies on the continuation principle and the construction of supersolution.

Theorem 2.5. *Let* $\max(p,l) \leqslant 1$. *Then problem* (1.1) *has global solutions for any coefficients* $c(x,t)$ *and* $k(x,y,t)$ *and any nonnegative initial data.*

Proof. Let T be any positive number. We construct a strict supersolution of (1.1) in Q_T. Suppose that $0<l<1$. By the conditions for $c(x,t)$ and $k(x,y,t)$, we have $c(x,t) \leqslant M$ and $k(x,y,t) \leqslant M$ in Q_T and $\partial\Omega \times Q_T$, respectively, where M is some positive constant. It is easy to verify that a function $v(t) = C\exp(\beta t)$ is a strict supersolution of (1.1) if $\beta \geqslant M$ and $C > \max\{\sup_{\overline{\Omega}} u_0(x), (M|\Omega|)^{1/(1-l)}, 1\}$. In the case that $l=1$, $v(t)$ is a strict supersolution of (1.1) if and only if

$$\int_\Omega k(x,y,t)\,dy < 1 \quad \text{for all } x \in \partial\Omega,\ 0 \leqslant t \leqslant T. \tag{2.3}$$

Therefore, when (2.3) is not valid we need another strict supersolution. Let us consider the eigenvalue problem given by

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{in } \Omega \text{ with } \varphi|_{\partial\Omega} = 0. \tag{2.4}$$

Denote its first eigenvalue λ_1 and the corresponding eigenfunction φ is chosen to satisfy that for some $0<\varepsilon<1$,

$$M\int_\Omega \frac{1}{\varphi(y)+\varepsilon}\,dy \leqslant 1. \tag{2.5}$$

Now, let $v(x,t)$ be defined as

$$v(x,t) = \frac{C\exp(\gamma t)}{\varphi(x)+\varepsilon}. \tag{2.6}$$

Then we have that

$$v_t - \Delta v - c(x,t)v^p \geqslant \gamma v - v\left(\frac{\lambda_1\varphi}{\varphi+\varepsilon} + \frac{2|\nabla\varphi|^2}{(\varphi+\varepsilon)^2}\right) - Mv \geqslant 0, \tag{2.7}$$

if

$$C \geqslant \sup_{\overline{\Omega}}(\varphi+\varepsilon) \quad \text{and} \quad \gamma \geqslant \lambda_1 + M + \sup_{\overline{\Omega}} \frac{2|\nabla\varphi|^2}{(\varphi+\varepsilon)^2}. \tag{2.8}$$

It is clear from (2.5)–(2.8) that $v(x,t)$ is a strict supersolution of problem (1.1) in Q_T if $C > \sup_{\overline{\Omega}} u_0(x) \sup_{\overline{\Omega}}(\varphi+\varepsilon)$ in addition. □

3. Blow-up and global existence for max(p, l) > 1

In this section we shall get some blow-up results for problem (1.1). The conditions for the existence of nonnegative global solutions of problem (1.1) with small initial data are also given.

Let φ be the eigenfunction of problem (2.4) corresponding to the first eigenvalue λ_1, which is chosen to satisfy that $\int_\Omega \varphi(x)\,dx = 1$. Now, we denote

$$\varphi_s = \sup_{\overline{\Omega}} \varphi(x) \tag{3.1}$$

and introduce an auxiliary function

$$w(t) = \int_\Omega u(x,t)\varphi(x)\,dx. \tag{3.2}$$

Further, we consider the Cauchy problem given by one of the following equations:

$$\begin{cases} v'(t) = -\lambda_1 v + c_0(t)v^p & \text{for } p > 1,\ 0 < l < 1, \\ v'(t) = -\lambda_1 v + k_0(t)v^l & \text{for } 0 < p < 1,\ l > 1, \\ v'(t) = -\lambda_1 v + c_0(t)v^p + k_0(t)v & \text{for } p > 1,\ l = 1, \\ v'(t) = -\lambda_1 v + c_0(t)v + k_0(t)v^l & \text{for } p = 1,\ l > 1, \\ v'(t) = -\lambda_1 v + c_0(t)v^p + k_0(t)v^l & \text{for } p > 1,\ l > 1, \end{cases} \tag{3.3}$$

with initial data

$$v(0) = w(0) = \int_\Omega u_0(x)\varphi(x)\,dx, \tag{3.4}$$

where

$$c_0(t) = \inf_{\overline{\Omega}} c(x,t) \quad \text{and} \quad k_0(t) = \frac{\lambda_1}{\varphi_s} \inf_{\partial\Omega\times\overline{\Omega}} k(x,y,t).$$

Note here that the solutions for any equation in (3.3) can be written in explicit form with the exception of the last one.

Theorem 3.1. *Let* $\max(p,l) > 1$ *and the Cauchy problem* (3.3), (3.4) *does not have a global solution. Then the solution of* (1.1) *blows up in a finite time.*

Proof. We suppose for the definiteness that $p > 1$ and $0 < l < 1$, since the proof of other cases is similar. Multiplying both sides of the first equation in (1.1) by $\varphi(x)$ and integrating over Ω, we have

$$w'(t) = \int_\Omega \big(\Delta u + c(x,t)u^p\big)\varphi(x)\,dx.$$

Then using (2.4), Green's identity and the equality $\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\,ds = -\lambda_1$, we obtain

$$\begin{aligned} w'(t) &= \int_\Omega \big(-\lambda_1 u + c(x,t)u^p\big)\varphi(x)\,dx - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\bigg(\int_\Omega k(x,y,t)u^l(y,t)\,dy\bigg)\,ds \\ &\geqslant \int_\Omega \big(-\lambda_1 u + c_0(t)u^p + k_0(t)u^l\big)\varphi(x)\,dx. \end{aligned} \tag{3.5}$$

Further, since $p > 1$, Jensen's inequality can be applied to (3.5) to get $w'(t) \geqslant -\lambda_1 w + c_0(t)w^p$. Thus, from the comparison principle for ordinary differential equations and by the hypothesis of this theorem, the conclusion follows. □

Remark 3.2. Note that from Theorem 3.1, we can get conditions for nonexistence of global solutions of (1.1) for all sufficiently large initial data. For example, it is not hard to verify that there are not global solutions in the case that $p > 1$, if

$$w(0) > \left[(p-1)\int_0^\infty c_0(t)\exp\big[(1-p)\lambda_1 t\big]\,dt\right]^{-\frac{1}{p-1}}$$

and in the case that $l > 1$, if

$$w(0) > \left[(l-1)\int_0^\infty k_0(t)\exp\big[(1-l)\lambda_1 t\big]\,dt\right]^{-\frac{1}{l-1}}.$$

In particular, there are not nontrivial nonnegative global solutions of (1.1) if $p > 1$ and

$$\int_0^\infty c_0(t)\exp\big[(1-p)\lambda_1 t\big]\,dt = \infty, \tag{3.6}$$

or if $l > 1$ and

$$\int_0^\infty k_0(t) \exp\bigl[(1-l)\lambda_1 t\bigr]\, dt = \infty. \tag{3.7}$$

As we shall show later in Theorem 3.5, the conditions (3.6) and (3.7) are exact in some sense.

The following Theorem 3.3 shows that for any nontrivial nonnegative initial data and under some conditions on the coefficients $c(x,t)$ and $k(x,y,t)$, there do not exist global solutions of (1.1). For the simplicity of our notation, we denote $P(u,t) \equiv -\lambda_1 u + c_0(t)u^p + k_0(t)u^l$ and let $\delta(t)$ be any nonnegative function such that

$$\int_0^\infty \delta(t)\, dt = \infty. \tag{3.8}$$

Theorem 3.3. *Let* $\max(p,l) > 1$ *and* $P(u,t) \geqslant \delta(t)u^{\max(p,l)}$ *for any* $u \geqslant 0$ *and* $t \geqslant 0$. *Then any solution of* (1.1) *with nontrivial initial datum blows up in a finite time.*

Proof. By similar arguments in Theorem 3.1, we have from (3.5) that

$$w'(t) \geqslant \int_\Omega P(u,t)\varphi(x)\, dx. \tag{3.9}$$

Suppose $\max(p,l) = p$, then we have $P(u,t) \geqslant \delta(t)u^p$. Otherwise, similar arguments hold replacing p by l. Substituting $\delta(t)u^p$ instead of $P(u,t)$ in (3.9) and using Jensen's inequality, we obtain that $w'(t) \geqslant \delta(t)w^p$, which in turn leads to

$$w(t) \geqslant \left[w(0)^{-(p-1)} - (p-1)\int_0^t \delta(\tau)\, d\tau \right]^{-\frac{1}{p-1}}.$$

Thus, from the definition of $\delta(t)$ in (3.8), $w(t)$ blows up in a finite time for any $w(0) > 0$. □

Remark 3.4. Assume that $c_0(t) \geqslant c_0$ and $k_0(t) \geqslant k_0$, where c_0 and k_0 are positive constants. Then we can show similarly that the conclusion in Theorem 3.3 holds with an assumption that for $\max(p,l) > 1$,

$$-\lambda_1 u + c_0 u^p + k_0 u^l > 0 \quad \text{for all } u > 0. \tag{3.10}$$

Indeed, it is not difficult to get from (3.10) that $P(u,t) \geqslant \varepsilon u^p$ if $p > 1$ and $P(u,t) \geqslant \varepsilon u^l$ if $l > 1$ for some positive ε and for all $u \geqslant 0$ and $t \geqslant 0$.

Now we shall show the existence of global solutions of problem (1.1) for sufficiently small initial data. Put $c_1(t) = \sup_{\overline{\Omega}} c(x,t)$. We suppose in the following statement that

$$\int_0^\infty c_1(t) \exp(-\gamma t)\, dt < \infty \quad \text{for some } \gamma < (p-1)\lambda_1, \tag{3.11}$$

and that for $x \in \partial\Omega$, $t \geqslant 0$ and some $A > 0$ and $\sigma < (l-1)\lambda_1$

$$\int_\Omega k(x,y,t)\, dy \leqslant A \exp(\sigma t), \tag{3.12}$$

where λ_1 is the first eigenvalue of problem (2.4).

Theorem 3.5. *Let* $\min(p,l) > 1$ *and* (3.11), (3.12) *hold. Then there exist nonnegative solutions of* (1.1) *with sufficiently small initial data, which are globally bounded.*

Proof. Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^n$ satisfying the property that $\Omega \Subset \widetilde{\Omega}$ and let $\tilde{\lambda}$ be the first eigenvalue of $-\Delta$ on $\widetilde{\Omega}$ with homogeneous Dirichlet boundary condition which satisfies the inequality given by

$$\max\left(\frac{\sigma}{l-1}, \frac{\gamma}{p-1}\right) \leqslant \tilde{\lambda} < \lambda_1. \tag{3.13}$$

Denote $\varphi(x)$ an eigenfunction corresponding to the eigenvalue $\tilde{\lambda}$, then it is obvious

$$\frac{\sup_{\widetilde{\Omega}} \varphi}{\inf_{\overline{\Omega}} \varphi} < a \tag{3.14}$$

for some $a > 1$. Now, choosing any ε which satisfies the inequality

$$0 < \varepsilon \leqslant \left(Aa^l\right)^{-\frac{1}{l-1}}, \tag{3.15}$$

and taking

$$\sup_{\widetilde{\Omega}} \varphi(x) = a\varepsilon, \tag{3.16}$$

we see from (3.14) that

$$\inf_{\partial\Omega} \varphi(x) > \varepsilon. \tag{3.17}$$

Next, for any $T > 0$, we construct a strict supersolution $v(x,t)$ of (1.1) in Q_T in such a form that $v(x,t) = \varphi(x) f(t)$ where

$$f(t) = \exp(-\tilde{\lambda}t)\left[B - (p-1)(a\varepsilon)^{p-1} \int_0^t c_1(\tau) \exp\left[-(p-1)\tilde{\lambda}\tau\right] d\tau\right]^{-\frac{1}{p-1}},$$

$$B = 1 + (p-1)(a\varepsilon)^{p-1} \int_0^\infty c_1(\tau) \exp\left[-(p-1)\tilde{\lambda}\tau\right] d\tau.$$

Then, since $f(t)$ is a solution of the equation

$$f' + \tilde{\lambda} f - (a\varepsilon)^{p-1} c_1(t) f^p = 0,$$

$v(x,t)$ satisfies the inequality $v_t \geqslant \Delta v + c(x,t)v^p$. By exploiting (3.11)–(3.13), (3.15)–(3.17) and the definition of $f(t)$, we obtain for $x \in \partial\Omega$ and $t > 0$, that

$$v(x,t) > \varepsilon f(t) \geqslant A(a\varepsilon)^l f(t) \geqslant \int_\Omega k(x,y,t)\varphi^l(y) f^l(t)\, dy.$$

Thus, by comparison principle, there exist global solutions of (1.1) for any nonnegative initial data such that $u_0(x) < B^{-\frac{1}{p-1}}\varphi(x)$. □

4. The case either $p = 1$ or $l = 1$

We can get more information for global existence and blow-up in a finite time of solutions for (1.1) when either $p = 1$ or $l = 1$. The following statement deals with the case that $p = 1$ with $l > 1$ and needs two new assumptions that

$$\int_0^\infty k_0(t) \exp\left[(l-1)(\beta - \lambda_1)t\right] dt = \infty, \tag{4.1}$$

$$\int_\Omega k(x,y,t)\, dy \leqslant M \exp\left[-(l-1)(\gamma - \lambda_1)t\right] \quad \text{for } x \in \partial\Omega,\ t \geqslant 0, \tag{4.2}$$

where $\beta \geqslant 0$, $M > 0$ and $\gamma > 0$.

Theorem 4.1. *Let $p = 1$ and $l > 1$. If $c(x,t) \geqslant \beta$ and $k(x,y,t)$ satisfies* (4.1)*, then any solution of* (1.1) *with nontrivial initial datum blows up in a finite time. On the other hand, if*

$$c(x,t) \leqslant c < \gamma \tag{4.3}$$

with some constant c and (4.2) *is valid, then there exist nonnegative solutions of* (1.1) *with sufficiently small initial data, which are globally bounded.*

Proof. To show the first part of this theorem, we note from (3.5) that $w'(t) \geqslant (\beta - \lambda_1)w + k_0(t)w^l$, where $w(t)$ is defined in (3.2). Changing the function $w(t) = v(t)\exp[(\beta - \lambda_1)t]$, we get $v'(t) \geqslant v^l k_0(t)\exp[(l-1)(\beta - \lambda_1)t]$. Thus the conclusion follows from the arguments of Theorem 3.3.

We now prove the latter part of this theorem. Let

$$\lambda_1 - (\gamma - c) < \tilde{\lambda} < \lambda_1. \tag{4.4}$$

As in the proof of Theorem 3.5 we consider eigenfunction $\varphi(x)$ which satisfies (3.16) and (3.17), where

$$0 < \varepsilon \leqslant \left[\frac{1}{a^l M}\right]^{\frac{1}{l-1}}. \tag{4.5}$$

Then $\bar{u} = \varphi(x)\exp[(c - \tilde{\lambda})t]$ is a strict supersolution of (1.1) in Q_T. Indeed,

$$\bar{u}_t \geqslant \Delta\bar{u} - c(x,t)\bar{u}.$$

On the other hand, by (3.16), (3.17), (4.2), (4.4) and (4.5), we obtain

$$\begin{aligned}\int_\Omega k(x,y,t)\bar{u}^l(y,t)\,dy &\leqslant (a\varepsilon)^l \exp\big[l(c-\tilde{\lambda})t\big] M \exp\big[-(l-1)(\gamma - \lambda_1)t\big] \\ &\leqslant \varepsilon \exp\big[(c-\tilde{\lambda})t\big] < \bar{u}(x,t) \quad \text{for } x \in \partial\Omega,\ t \geqslant 0.\end{aligned}$$

Thus, by comparison principle, there exist global solutions of (1.1) for any nonnegative initial data such that $u_0(x) < \varphi(x)$. □

Now we demonstrate the exactness of Theorem 4.1 in two remarks.

Remark 4.2. It is not difficult to deduce from Theorem 4.1 an optimality of exponents in exponential functions of the conditions (4.1) and (4.2).

Remark 4.3. To explain a presence of c in (4.3) we put $\gamma = \lambda_1$ and suppose that $k(x,y,t) \equiv M_0 > 0$ and $c(x,t) = \lambda_1 - \delta(t)$, where a nonnegative function $\delta(t) \leqslant \lambda_1$ will be specified later. Then, by (3.5) we have that $w'(t) \geqslant -\delta(t)w(t) + M_1 w^l$, where $M_1 = \lambda_1 M_0/\varphi_s$ and $w(t)$ is defined in (3.2). Now we consider the Cauchy problem

$$g'(t) = -\delta(t)g(t) + M_1 g^l \quad \text{for } g(0) = \alpha > 0, \tag{4.6}$$

where the solution $g(t)$ of (4.6) can be written as

$$g(t) = \left\{\alpha^{-(l-1)}\exp\left[(l-1)\int_0^t \delta(\tau)\,d\tau\right] - (l-1)M_1\int_0^t \exp\left[(l-1)\int_\tau^t \delta(s)\,ds\right]d\tau\right\}^{-\frac{1}{l-1}}.$$

It is obvious that $g(t)$ blows up in a finite time for all $\alpha > 0$ if, for example,

$$\int_0^\infty \delta(t)\,dt < \infty. \tag{4.7}$$

Thus, if $w(0) \geqslant \alpha$, then by comparison principle for the Cauchy problem (4.6) $w(t) \geqslant g(t)$, which in turn implies that for any $w(0) > 0$ function $w(t)$ exists only finite time if $\delta(t)$ satisfies (4.7). Therefore, any solution of (1.1) with nontrivial nonnegative initial datum blows up in a finite time. Hence, under the condition (4.2) and the inequality $c(x,t) < \gamma$, problem (1.1) cannot have nontrivial global solutions.

The following Theorem 4.4 is the case when $l = 1$ and $p > 1$. Any solution of (1.1) blows up with the conditions that

$$\int_0^\infty c_0(t)\,dt = \infty \quad \text{and} \quad \int_\Omega k(x,y,t)\,dy \geqslant 1 \quad \text{for } x \in \partial\Omega,\ t \geqslant 0, \tag{4.8}$$

and conversely there exist global solutions of (1.1) with small initial data and the conditions that

$$c(x,t) \leqslant M \quad \text{for } x \in \overline{\Omega},\ t \geqslant 0, \quad \text{and} \quad \int_\Omega k(x,y,t)\,dy \leqslant K < 1 \quad \text{for } x \in \partial\Omega,\ t \geqslant 0. \tag{4.9}$$

Theorem 4.4. *Let $l = 1$ and $p > 1$. If $c(x,t)$ and $k(x,y,t)$ satisfy* (4.8)*, then any solution of* (1.1) *with nontrivial nonnegative initial datum blows up in a finite time. If* (4.9) *holds, then there exist nonnegative solutions of* (1.1) *with sufficiently small initial data, which are globally bounded.*

Proof. We suppose firstly that (4.8) holds. Let $t_0 > 0$ and $u(x,t)$ be a solution of (1.1). By Lemma 2.3 there exists $\varepsilon > 0$ such that $u(x,t_0) \geqslant \varepsilon$ for any $x \in \overline{\Omega}$. It is not difficult to verify that

$$h(t) = \left[\varepsilon^{-(p-1)} - (p-1)\int_{t_0}^t c_0(\tau)\,d\tau\right]^{-\frac{1}{p-1}}$$

is the subsolution of the problem (1.1) in $Q_T \cap \{t > t_0\}$ for any $t_0 < T < T_\star$, where $T_\star$ is determined from the equality

$$\int_{t_0}^{T_\star} c_0(\tau)\,d\tau = \frac{1}{(p-1)\varepsilon^{(p-1)}}.$$

Because $h(t)$ blows up in a finite time we get conclusion by comparison principle.

To prove the second part of this theorem we can construct domain $\widetilde{\Omega}$ such that $a \leqslant 1/K$ in (3.14) and then apply the arguments of Theorem 4.1 with

$$0 < \varepsilon \leqslant \frac{1}{a} \inf_{x\in\overline{\Omega},\,t\geqslant 0}\left[\frac{\tilde{\lambda}}{c(x,t)}\right]^{\frac{1}{p-1}}. \qquad \square$$

Remark 4.5. We note here an importance of divergence of the integral in (4.8) for conclusion of the first part of Theorem 4.4. Assume that

$$c(x,t) = c_0(t), \quad \int_0^\infty c_0(t)\,dt < \infty \quad \text{and} \quad \int_\Omega k(x,y,t)\,dy = 1 \quad \text{for } x \in \partial\Omega,\ t \geqslant 0.$$

Then the solutions of (1.1) exist globally for sufficiently small initial data. Indeed, we can show directly that the function

$$y(t) = \left[\beta^{-(p-1)} - (p-1)\int_0^t c_0(\tau)\,d\tau\right]^{-\frac{1}{p-1}}$$

is a supersolution of (1.1) if $u_0(x) < \beta$ and $0 < \beta < [(p-1)\int_0^\infty c_0(t)\,dt]^{-\frac{1}{p-1}}$.

To show an optimality of the second condition in (4.9), we prove that for some functions $c(x,t)$ and $k(x,y,t)$ such that

$$\int_0^\infty c_0(t)\,dt = \infty \quad \text{and} \quad 0 < \int_\Omega k(x,y,t)\,dy < 1 \quad \text{for } x \in \partial\Omega,\ t \geqslant 0, \tag{4.10}$$

any solution of (1.1) with nontrivial nonnegative initial datum blows up in a finite time. Note that there exists wide class functions $c(x,t)$ which satisfy both first conditions in (4.9) and (4.10). However, if (4.9) is valid, then for sufficiently small initial data there exist nonnegative globally bounded solutions of (1.1).

Theorem 4.6. *Let $l=1$ and $p>1$. Then there exist functions $c(x,t)$ and $k(x,y,t)$ with properties* (4.10) *such that any solution of* (1.1) *with nontrivial nonnegative initial datum blows up in a finite time.*

Proof. To prove theorem we construct a subsolution of (1.1) which blows up in a finite time. By Lemma 2.3 for any $t_0>0$ there exists $\varepsilon>0$ such that

$$u(x,t_0) > \varepsilon \quad \text{for } x \in \overline{\Omega}. \tag{4.11}$$

Let $\alpha(t)$ be a smooth function which satisfies the following relations:

$$\alpha(0) = \frac{1}{\varphi_s}, \quad \alpha(t) > 0, \quad \alpha'(t) \leqslant 0 \quad \text{and} \quad \int_0^\infty \alpha(t)\,dt < \infty, \tag{4.12}$$

where φ_s was defined in (3.1). Put

$$k(x,y,t) \equiv k(t) = \frac{1}{\alpha(t) + |\Omega|}, \tag{4.13}$$

then obviously,

$$\int_\Omega k(x,y,t)\,dy < 1 \quad \text{and} \quad \int_\Omega k(x,y,t)\,dy \to 1 \quad \text{as } t \to \infty.$$

Now, let $f(t)$ be the solution of the following problem:

$$\begin{cases} f'(t) = -\lambda_1\varphi_s\alpha(t)f(t) + c_0(t)f^p(t) & \text{for } t > t_0, \\ f(t_0) = \frac{\varepsilon}{2}. \end{cases} \tag{4.14}$$

Then $f(t)$ can be written in an explicit form that

$$f(t) = \left\{ \left(\frac{2}{\varepsilon}\right)^{p-1} \exp\left[(p-1)\lambda_1\varphi_s \int_{t_0}^t \alpha(\tau)\,d\tau\right] - (p-1)\int_{t_0}^t c_0(\tau)\exp\left[(p-1)\lambda_1\varphi_s\int_\tau^t \alpha(s)\,ds\right]d\tau \right\}^{-\frac{1}{p-1}},$$

and thus $f(t)$ exists only for $t < T_\varepsilon < \infty$. Now, we set

$$\underline{u}(x,t) = f(t)\big[\alpha(t)\varphi(x) + 1\big], \tag{4.15}$$

where $\varphi(x)$ satisfies (2.4) with $\int_\Omega \varphi(x)\,dx = 1$. Then $\underline{u}(x,t)$ is a subsolution of problem (1.1) in $Q_T \cap \{t > t_0\}$ for any $t_0 < T < T_\varepsilon$. Indeed, exploiting (2.4), (4.12), (4.14) and (4.15), we get

$$\underline{u}_t - \Delta\underline{u} - c(x,t)\underline{u}^p \leqslant f'(\alpha\varphi + 1) + \lambda_1\alpha\varphi f - c_0(t)(\alpha\varphi+1)^p f^p \leqslant 0.$$

Further, by (4.13) and (4.15), we have

$$\underline{u}(x,t) = \int_\Omega k(x,y,t)\underline{u}(y,t)\,dy \quad \text{for } x \in \partial\Omega,\ t > 0,$$

and, moreover, using (4.11), (4.12), (4.14) and (4.15), we find that $\underline{u}(x,t_0) < u(x,t_0)$. Thus, the conclusion follows from Theorem 2.4. $\square$

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